10. The equation of a curve is given by $y = \frac{2x + 6}{x - 1}$.

(a) If $\frac{dy}{dx} = \frac{k}{(x-1)^2}$, where k is a constant, find the value of k.

(b) The tangents to the curve at the points A and C, where A lies in the first quadrant, are parallel to the line $3x + 6y - 8 = 0$. Find the coordinates of A and C.

(c) Find the equations of the tangent and the normal to the curve at A.

(d) Find the equations of the tangent and the normal to the curve at C.

(e) What type of quadrilateral is formed by the tangents and the normals at A and C? Find the area of the quadrilateral.

13.2 Rates Of Change

The derivative of a function $y = f(x)$ can be interpreted as the rate of change of the variable y with respect to the variable x. Let us explore this interpretation in the following Class Activity.

CLASS ACTIVITY 1

Objective: To understand that the derivative of a function can be interpreted as the rate of change of a variable.

A car accelerates from rest along a straight road. Its displacement y m from the starting point at time t seconds is given by $y = f(t) = \frac{3}{2}t^2$. Let us find the velocity of the car when $t = 5$.

1. Copy and complete the following table, where $\delta y = f(5 + \delta t) - f(5)$.

δt	1	0.5	0.2	0.1	0.001
$5 + \delta t$	6	5.5			
$f(5 + \delta t)$	54	45.375			
δy					
$\frac{\delta y}{\delta t}$					

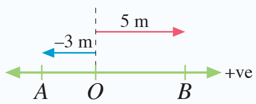
2. In **Question 1**, what does $\frac{\delta y}{\delta t}$ represent?

3. Suggest the value of $\lim_{\delta t \to 0} \frac{\delta y}{\delta t}$ in **Question 1**. What does it represent?

4. Express $\lim_{\delta t \to 0} \frac{f(t + \delta t) - f(t)}{\delta t}$ in differentiation notation.

REMARK

Displacement of an object refers to the distance between the initial position of the object and the final position of the object in a specified direction.



In the above diagram, if we regard the direction from O to B as positive, the displacement from O to A is –3 m and that from O to B is 5 m.

In Class Activity 1,

$$\frac{\delta y}{\delta t} = \frac{f(t + \delta t) - f(t)}{\delta t}, \text{ where } t = 5,$$

represents the average velocity of the car during the interval between $t = 5$ and $t = 5 + \delta t$. When δt approaches 0,

$$\lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \lim_{\delta t \to 0} \frac{f(t + \delta t) - f(t)}{\delta t} \text{ .... } (*)$$

becomes the **instantaneous velocity** of the car at $t = 5$.

Remark

Instantaneous velocity refers to the velocity at a certain point in time.

Note that the expression (∗) resembles the definition of the derivative of $y = f(x)$:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

Hence, we can say that:

The derivative of $y = f(t)$, $\frac{dy}{dt} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t}$, is the rate of change of y with respect to t.

This concept is widely used in Physics and Economics. In particular, when $y = f(t)$ is the displacement function at time t, $\frac{dy}{dt}$ gives the instantaneous velocity at time t.

Example 5 (Rate of change in a clinical trial)

After taking a dosage of a certain medicine for t hours, the amount, W mg, of the medicine remaining in the body of a patient is given by

$$W = \frac{50}{1 + t^2}.$$

(a) Find the amount of the medicine remaining in the patient's body when
(i) $t = 0$, **(ii)** $t = 2$.
(b) Find the rate of change of W at $t = 2$.

Solution

(a) (i) $W = \frac{50}{1 + t^2}$

When $t = 0$, $W = \frac{50}{1 + 0^2}$
$= 50$

i.e. when $t = 0$, the amount of the medicine remaining in the patient's body is 50 mg.

(ii) When $t = 2$, $W = \frac{50}{1 + 2^2}$
$= 10$

i.e. when $t = 2$, the amount of the medicine remaining in the patient's body is 10 mg.

(b) Differentiate W with respect to t,

$$\frac{dW}{dt} = \frac{d}{dt}[50(1 + t^2)^{-1}]$$
$$= 50(-1)(1 + t^2)^{-2}(2t) \quad \text{Chain rule}$$
$$= -100t(1 + t^2)^{-2}$$
$$\left.\frac{dW}{dt}\right|_{t=2} = -100(2)(1 + 2^2)^{-2}$$
$$= -8$$

i.e. the rate of change of W at $t = 2$ is –8 mg/h.

Note: $\frac{dW}{dt} < 0$ means that the amount of the medicine remaining in the body is decreasing.

Try It 5! Water is dripping into a bowl from a leaking tap such that the depth, h cm, of the water in the bowl at time t seconds is given by $h = \sqrt{1 + 2t}$.

(a) Find the depth of the water when
(i) $t = 0$, **(ii)** $t = 4$.

(b) Find the rate of change of h at $t = 4$.

Example 6 (Motion of particle)

A cork is bobbing up and down on the water in a container. Its vertical displacement, y cm, from a reference level at time t seconds is given by $y = 3 \sin 2t$. Find the velocity of the cork when

(a) $t = \frac{\pi}{6}$, **(b)** $t = \frac{\pi}{2}$.

REMARK
This type of oscillation motion is called the simple harmonic motion.

Solution

(a)
$$y = 3 \sin 2t$$
$$\frac{dy}{dt} = 3(2) \cos 2t$$
$$= 6 \cos 2t$$
$$\left.\frac{dy}{dt}\right|_{t=\frac{\pi}{6}} = 6 \cos 2\left(\frac{\pi}{6}\right)$$
$$= 3$$

i.e. when $t = \frac{\pi}{6}$, the velocity of the cork is 3 cm/s.

(b)
$$\left.\frac{dy}{dt}\right|_{t=\frac{\pi}{2}} = 6 \cos 2\left(\frac{\pi}{2}\right)$$
$$= -6$$

i.e. when $t = \frac{\pi}{2}$, the velocity of the cork is –6 cm/s.

Note:
1. When the velocity is positive, it means that the displacement is increasing at that moment.
2. When the velocity is negative, it means that the displacement is decreasing at that moment.

Try It 6! The displacement, x cm, of a particle from a fixed point at time t seconds is given by $x = 8 \cos 5t$. Find its velocity when

(a) $t = \frac{\pi}{15}$,

(b) $t = \pi$.

Exercise 13.2

Level 1

1. Eliza jogs along a track. Her displacement, y m, from a starting point at time t seconds is given by $y = 3t$. Find her jogging velocity when $t = 5$.

2. When a ball is dropped from the top of a building, its height, H m, from the ground at time t seconds is given by $H = 180 - 5t^2$. Find the velocity of the ball when

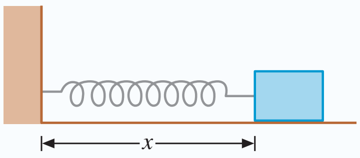
(a) $t = 2$,

(b) the ball hits the ground.

3. In the diagram, the length, x cm, of a spring at time t seconds is given by $x = 20 + 3 \sin 4t$.

(a) Find the shortest length of the spring.

(b) Find the rate of change of x when $t = \frac{\pi}{8}$.



4. In an animation illustrating the idea of rate of change, the area, A cm^2, of an image in the animation at time t seconds is given by $A = 2t^3 + 5t$. Find the rate of change of the area of the image at $t = 3$.

5. The volume, V cm^3, of a piece of melting ice at time t seconds is given by $V = \frac{60}{1 + t}$.

(a) Find the initial volume of the ice.

(b) Find the rate of change of the volume of the ice at $t = 4$.

6. In a biology experiment, the height, H cm, of a plant at time t months after the first observation is modelled by

$$H = 60 + \sqrt{8t}.$$

(a) Find the height of the plant when

(i) $t = 0$,

(ii) $t = 2$.

(b) Find the rate of increase of the height of the plant when $t = 2$.

Level 2

7. On a coordinate plane, a particle moves along the x-axis. Its x-coordinate at time t seconds is given by $x = \frac{1}{3}t^3 - 5t^2 + 9t$.

(a) Find the rate of change of x with respect to t when $t = 4$.

(b) Find the values of t when the particle is momentarily at rest.

8. In a chemical process, the mass, M g, of a product at time t minutes is given by $M = \frac{30t}{3t + 1}$.

(a) Find the average rate of change of the mass during the period

(i) from $t = 2$ to $t = 3$,

(ii) from $t = 2$ to $t = 2.1$.

(b) Find the rate of change of the mass of the product when $t = 2$.

Level 3

9. A stone is thrown vertically upwards. Its height, h m, above the ground at time t seconds is given by

$$h = 20t - 5t^2,$$

where $0 \leqslant t \leqslant T$, and T is the time of flight of the stone.

(a) Find the value of T.

(b) Draw the graph of h against t.

(c) Find the interval of the time t for which the velocity is positive.