## 1.6 Division

Division of whole numbers is defined in terms of multiplication using the idea of a "missing factor".

**Definition 6.1.** Division *is defined by missing factors: the number* $56 \div 8$ *is the missing factor in* $\underline{\hspace{2em}} \times 8 = 56$.

By this definition $56 \div 8 \times 8 = 56$, so if we divide by 8 and then multiply by 8 we get our original number back. Thus multiplication and division are "opposite operations" in exactly the same way that addition and subtraction are.

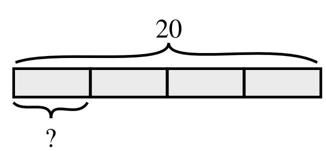
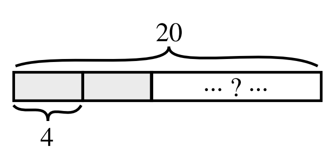
The numbers in a division problem can be unambiguously identified using the terms dividend, divisor, and quotient:

$$\underbrace{56}_{\text{dividend}} \div \underbrace{8}_{\text{divisor}} = \underbrace{7}_{\text{quotient.}}$$

As with multiplication, division word problems use both the set and measurement models. But division is more interesting because there are *two distinct interpretations of division*. For example, we can explain $20 \div 4$ using the "missing factor" approach: $20 \div 4$ is the solution to either

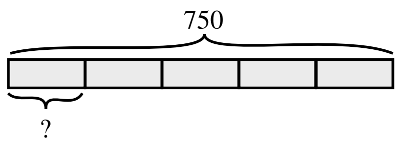
$$20 = 4 \times \underline{\hspace{2em}} \qquad \text{or} \qquad 20 = \underline{\hspace{2em}} \times 4.$$

Because multiplication is commutative, these two equations have the same numerical answer, namely 5. But they have different interpretations. The first asks "20 is 4 groups of what unit?", while the second asks "20 is how many groups of 4 units?". These two interpretations of division have names, and each can be neatly visualized as *bar diagrams*.

| Interpretation | Interpretive question | Diagram |
|---|---|---|
| Partitive division: | 20 is 4 groups of what unit? |   |
| Measurement division: | 20 is how many groups of 4? |   |

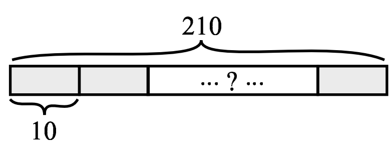
When we know the original amount and the *number of parts*, we use partitive division to find the size of each part. On the other hand, when we know the original amount and the *size or measure of one part*, we use measurement division to find the number of parts. These two interpretations are easily distinguished in word problems. Primary Mathematics 2A uses simple word problems and illustrations to describe both interpretations and show why they are different.

**Example 6.2.** *5 packets of coffee beans weigh 750 g. How much does each packet weigh?*




This problem asks "750 g is 5 packets of what size packet?". This is partitive division — the number of groups (packets) is specified and we must find the size of each. One can also think of distributing the 750 grams equally among 5 packets (or 5 people), and for that reason partitive division is sometimes called *sharing division.*


**Example 6.3.** *Sarah made 210 cupcakes. She put them into boxes of 10 each. How many boxes of cupcakes were there?*




This question asks "210 is how many boxes of 10 cupcakes?" This is measurement division — the unit (10 cupcakes) is specified and we must find the number of units.


Division word problems use one or the other of these interpretations. To identify which, draw the bar diagram, or note whether we are asked for the size of each part parts (partitive) or the number of parts (measurement). On the other hand, straight numerical problems, such as $35 \div 5$, have no built-in interpretation. In order to create a word problem illustrating $35 \div 5$ *one of the interpretations has to be chosen.*

Creating a division word problem for $a \div b$ involves three steps:

1. Make a choice: partitive or measurement division.
2. Think of the corresponding diagram or question ("$X$ is $Y$ groups of what?" or "$X$ is how many groups of $Y$?"). This provides a 'skeleton' of a word problem.
3. Flesh this out into a real-world problem by answering "groups or amounts of what?".

**Example 6.4.** — Here are two ways to create a word problem for $35 \div 5$ following the steps above.

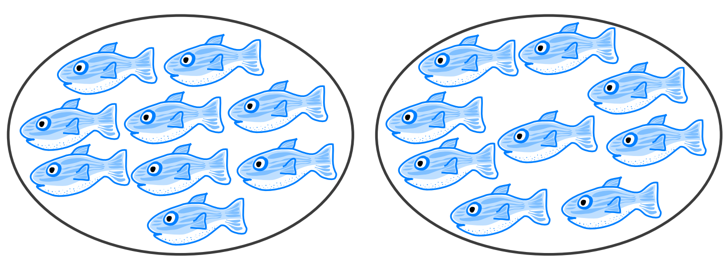
(a) Choosing partitive division, the question is "35 is 5 groups of what?". Thinking of weight and food leads to the word problem "If 5 candy bars weigh 35 ounces, how much does each one weigh?"

(b) Choosing measurement division, the question is "35 is how many 5s?". Thinking of money and clothes leads to the word problem "T-shirts cost $5 each. How many can you buy for $35?".

Making up word problems is a skill acquired through practice. It is discussed in Chapter 2.

### A Teaching Sequence for Division

The Primary Mathematics textbooks introduce division in second grade, first as partitive division and then as measurement division. Pictures are very important in the beginning. Shortly after division is introduced, four-fact families with the associated pictures are presented.



| | | | |
|---|---|---|---|
| $9 \times 2 =$ | ___ | $2 \times 9 =$ | ___ |
| $18 \div 2 =$ | ___ | $18 \div 9 =$ | ___ |

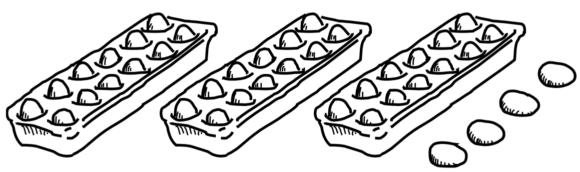
This encourages students to do division by recalling known multiplication facts. From that point on, multiplication and division are taught in tandem. For instance, students learn the multiples of 2 and 3, then in the next section learn how to divide by 2 and 3.

In any curriculum, it is important that students see many examples of both partitive and measurement division — both are needed for 'real-life' applications and for understanding later concepts (for example, division of fractions). Students who see a balanced mix of the two types of division problems will more quickly learn to recognize division in its different guises and to view division as a single, simple, mathematical operation.

*Why distinguish partitive and measurement division?* This distinction is not something students need to know, but it is something that can make you a better teacher. A key point of this section is that students must come to associate the operation of division with completely two different types of word problems. That difference is not apparent to adults, who long ago learned to instantly associate both with division; it requires effort to "undo" that automatic association. But teachers who learn to distinguish partitive and measurement division are better able to understand their students' thinking. They are in a better position to insure that their students see an appropriate mix of division problems, and are better prepared for making up word problems for division.

Thus far in this section, as in the Primary Mathematics textbooks through the middle of Grade 3, all division problems have had whole number answers — there have been no remainders. The introduction of remainders is the next stage in teaching division. Remainders appear naturally in both set and measurement models as the two examples below show.

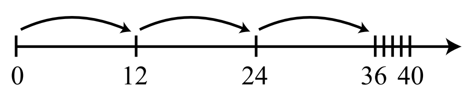
40 eggs is how many dozen?



3 groups with 4 left over.

40 ÷ 12 = 3 R 4

How many feet in 40 inches?




Three feet with 4 inches left.

40 ÷ 12 = 3 R 4

Underlying all divisions-with-remainder problems is a single mathematical fact, called the Quotient–Remainder Theorem. The theorem generalizes the missing factor definition of division. For the example $40 \div 12$, it says that there are unique whole numbers which fill in the blanks

$$40 = (12 \times \underbrace{\quad}_{\text{quotient}}) + \underbrace{\quad}_{\text{remainder}}$$

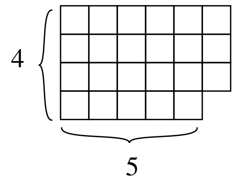
in such a way that the remainder is less than 12. The proof is exactly as in the pictures above: starting at 40 we can repeatedly subtract 12 until we reach a whole number less than 12. Of course, there is nothing special about the numbers 40 and 12. The theorem holds if we replace the number 40 by an arbitrary whole number called '$A$' and similarly replace 12 by a whole number '$k$'. If $A$ is a multiple of $k$ then the quotient $A \div k$ is the number $q$ which satisfies $A = k \times q$. In general, there is a remainder $r$ and that remainder is less than $k$. Thus

**Theorem 6.5 (Quotient–Remainder Theorem).** *For any two whole numbers $A$ and $k$ with $k \neq 0$ there are unique whole numbers $q$ (the quotient) and $r$ (the remainder) such that*

$$A = (k \times q) + r$$

*and $0 \leq r < k$.*

**Example 6.6.** *The Quotient–Remainder Theorem for* $23 \div 4$ *can be illustrated by a "rectangular array with remainder"*




$$23 = (4 \times 5) + 3, \textit{ or}$$

$$23 \div 4 = 5 \textit{ R3.}$$

Finally, there is one minor point that inevitably comes up in discussions of division: division by 0. Most textbooks state that division by 0 is 'undefined'. That statement does not mean that dividing by zero is against some rule or law. Rather, it means that *there is no answer that makes sense.* Understanding that point requires considering two separate cases.

*Case 1*: If $10 \div 0$ were equal to some number, it would be the missing factor in $\_\_\_ \times 0 = 10$ ("how many groups of size 0 make 10?"). There is no such number! Thus $10 \div 0$ does not specify a number.

*Case 2*: $0 \div 0$ is also undefined, but for a different reason. Solving $0 \div 0 = \_\_\_$ is the same as solving $0 = 0 \times \_\_\_$. But *any* number can fit in the blank. Thus the division expression $0 \div 0$ does not represent one particular number, as all other division expressions do.

In the first case division by zero does not specify *any* number, while in the second case it specifies *every* number. The phrase "division by 0 is undefined" is the standard way of expressing both these cases at once.

**Homework Set 6**

1. Identify whether the following problems are using measurement or partitive division (if in doubt, draw a bar diagram).
   (a) Jim tied 30 sticks into 3 equal bundles. How many sticks were there in each bundle?
   (b) 24 balls are packed into boxes of 6. How many boxes are there?
   (c) Mr. Lin tied 192 books into bundles of 6 each. How many bundles were there?
   (d) 6 children shared 84 balloons equally. How many balloons did each child get?
   (e) Jill bought 8 m of cloth for \$96. Find the cost of 1 m of cloth.
   (f) We drove 1280 miles from Michigan to Florida in 4 days. What was our average distance per day?

2. Identify whether the following problems are using measurement or partitive division.
   (a) Problems 10 and 11 on page 43 of Primary Math 3A.
   (b) Problems 6 – 10 on page 65 of Primary Math 3A.
   (c) Problems 7 – 9 on page 35 of Primary Math 4A.

3. Illustrate the following with a bar diagram and solve the problem.
   (a) measurement division for $56 \div 8$.
   (b) partitive division for $132 \div 4$.
   (c) measurement division for $4096 \div 512$.
   (d) partitive division for $512 \div 8$.
   (e) measurement division for $140 \div 20$.
   (f) measurement division for $143 \div 21$.

4. Make up a word problem for the following using the procedure of Example 6.4.

(a) measurement division for $84 \div 21$.
(b) partitive division for $91 \div 5$.
(c) measurement division for $143 \div 21$.

5. Illustrate the Quotient–Remainder Theorem for the following cases.

   (a) Number line model for $59 \div 10$.
   (b) Set model for $14 \div 4$.
   (c) Bar diagram (using measurement division) for $91 \div 16$.

6. One might guess that the properties of multiplication also hold for division, in which case we would have:

   (a) Commutative: $a \div b = b \div a$.
   (b) Associativity: $(a \div b) \div c = a \div (b \div c)$.
   (c) Distributivity: $a \div (b + c) = (a \div b) + (a \div c)$.

   whenever $a$, $b$, and $c$ are whole numbers. By choosing specific values of the numbers $a$, $b$, and $c$, give examples (other than dividing by zero) showing that each of these three "properties" is false.

## 1.7 Addendum on Classroom Practice

This course focuses on mathematics. But in addition to knowing subject matter, teachers must also know what works and does not work in the classroom. Thus this course is only a first step toward becoming an effective mathematics teacher. Your education courses will build on this course, giving you the knowledge and classroom skills needed to teach mathematics effectively, topic by topic.

The distinction between what is mathematics and what is teaching methodology can be confusing. This course contains many discussions (of models, component skills, teaching sequences, potential student errors, etc.) which might at first seem to be pedagogy, not mathematics. But those topics are instances where *the logic of the mathematics dictates the order and manner in which mathematics is developed in the classroom.* Understanding such topics is essentially a matter of understanding the mathematics.

This supplementary section gives some ideas about what lies beyond this course. It lists and comments on some of the many aspects of classroom practice that will be covered in your teacher education courses. Of course, classroom practice varies from grade to grade, subject to subject, and topic to topic. The teaching techniques that work best for teaching division to second graders are different from those best for teaching fractions to fourth graders. To keep the discussion concrete we will give examples related only to teaching counting, leaving it to you to see how the same themes apply to other topics in mathematics.

**Lesson Planning.** As a teacher, you will plan and prepare lessons every day. The quality of that preparation will be a large factor in your success as a teacher. But there will be no one in the classroom to help you and the time available for preparation will be limited. You will get essentially no guidance or support from your school administrators. Your teacher's manual is likely to be woefully inadequate. You will be continually scrambling to produce, from somewhere, supplementary material for the best students, for the weakest students, and for the class as a